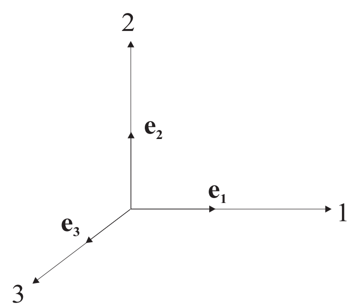
Handout # 3:
Cartesian Tensors

Vectors in Cartesian Coordinates

Coordinates system E with unit vectors

$$\boldsymbol{e_1}, \quad \boldsymbol{e_2}, \quad \boldsymbol{e_3} \quad \Rightarrow \quad \boldsymbol{e_i} \quad \text{(vector in } i\text{-direction)}$$



Since unit vectors are orthogonal basis

$$\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \delta_{ij}$$

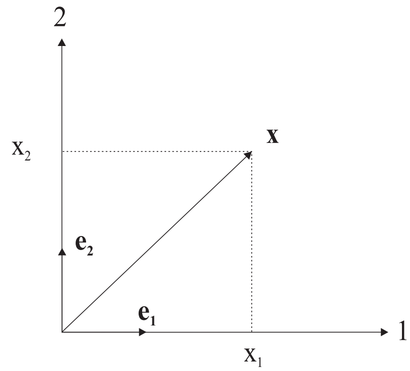
with Kronecker delta δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Index substitution:

$$\delta_{ij}\, x_i = x_j$$

Coordinate System in 2D



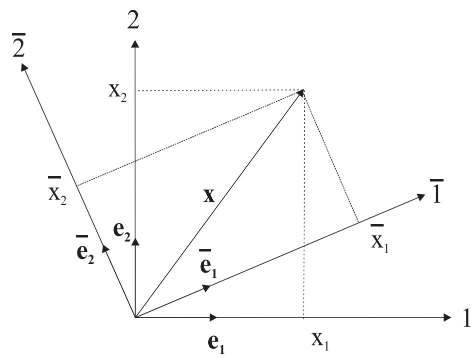
Any vector $\boldsymbol{x}$ can be represented in E by

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{e_1}\,x_1 + \boldsymbol{e_2}\,x_2 + \boldsymbol{e_3}\,x_3 \\ &= \boldsymbol{e_i}\,x_i \qquad \text{(dummy index implies summation)}\end{aligned}$$

Definition of Tensor

- **Tensor consists of tensor components (**_e.g._ x_i**) and coordinate system (**_e.g._ E**)**
- Tensor remains unchanged if expressed in coordinate system $\overline{E}$ obtained by axes rotation and reflection of system E

Consider two coordinate systems E and $\overline{E}$:



$$\boldsymbol{x} \;=\; \boldsymbol{e_i}\, x_i = \overline{\boldsymbol{e_i}}\, \overline{x_i}$$

$$\begin{aligned} \boldsymbol{e_k} \cdot \boldsymbol{e_i}\, x_i \;&=\; \boldsymbol{e_k} \cdot \overline{\boldsymbol{e_i}}\, \overline{x_i} \\ \delta_{ki}\, x_i \;=\; x_k \;&=\; a_{ki}\, \overline{x_i} \end{aligned}$$

where

$$a_{ki} \;=\; \boldsymbol{e_k} \cdot \overline{\boldsymbol{e_i}} \;=\; |\boldsymbol{e_k}||\overline{\boldsymbol{e_i}}| \cos \alpha_{ki} \;=\; \cos \alpha_{ki}$$

since $|\boldsymbol{e_k}| = |\overline{\boldsymbol{e_i}}| \equiv 1$.

If transformation can be written as

$$x_i \;=\; a_{ij}\, \overline{x_j}$$

then $\boldsymbol{x}$ is a tensor.

Second order tensor

$$\boldsymbol{b} \;=\; \boldsymbol{e_i}\, \boldsymbol{e_j}\, b_{ij} \;=\; \overline{\boldsymbol{e_i}}\, \overline{\boldsymbol{e_j}}\, \overline{b_{ij}}$$

Tensor operations

- Tensor product

$$\boldsymbol{u}\,\boldsymbol{b} \;=\; \boldsymbol{e_i}\,u_i\,\boldsymbol{e_j}\,\boldsymbol{e_k}\,b_{jk} \;=\; \boldsymbol{e_i}\,\boldsymbol{e_j}\,\boldsymbol{e_k}\,u_i\,b_{jk} \qquad \text{(order N + M)}$$

- Inner product

$$\boldsymbol{u}\cdot\boldsymbol{b} \;=\; \boldsymbol{e_i}\,u_i\cdot\boldsymbol{e_j}\,\boldsymbol{e_k}\,b_{jk} \;=\; \delta_{ij}\,\boldsymbol{e_k}\,u_i b_{jk} \;=\; \boldsymbol{e_k}\,u_j\,b_{jk} \qquad \text{(order N + M - 2)}$$

- Gradient

$$\nabla \;\equiv\; \boldsymbol{e_i}\,\frac{\partial}{\partial x_i}$$

$$\nabla\boldsymbol{b} \;=\; \boldsymbol{e_i}\,\frac{\partial}{\partial x_i}\,(\,\boldsymbol{e_j}\,\boldsymbol{e_k}\,b_{jk}\,) = \boldsymbol{e_i}\,\boldsymbol{e_j}\,\boldsymbol{e_k}\,\frac{\partial b_{jk}}{\partial x_i} \qquad \text{(order 1 + M)}$$

- Divergence

$$\nabla\cdot\boldsymbol{b} \;=\; \boldsymbol{e_i}\cdot\frac{\partial}{\partial x_i}\,(\,\boldsymbol{e_j}\,\boldsymbol{e_k}\,b_{jk}\,) = \delta_{ij}\,\boldsymbol{e_k}\,\frac{\partial b_{jk}}{\partial x_i} \;=\; \boldsymbol{e_k}\,\frac{\partial b_{ik}}{\partial x_i} \qquad \text{(order M-1)}$$

- Cross product

$$\boldsymbol{u}\times\boldsymbol{v} \;=\; \det\begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Alternating symbol:

$$\epsilon_{ijk} \;=\; \begin{cases} 1 & \text{if } ijk \text{ are cyclic} \qquad (123,\ 231,\ 312) \\ -1 & \text{if } ijk \text{ are anti-cyclic} \quad (321,\ 213,\ 132) \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow\ \boldsymbol{u}\times\boldsymbol{v} = \epsilon_{ijk}\,\boldsymbol{e_i}\,u_j\,v_k \quad \text{and} \quad \boldsymbol{\omega} \;=\; \nabla\times\boldsymbol{u} \;=\; \epsilon_{ijk}\,\boldsymbol{e_i}\,\frac{\partial u_k}{\partial x_j}$$

Note: ϵ_{ijk} is not a tensor, hence the cross product is not a tensor operation.

Momentum equations in direct and suffix notation:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \underbrace{\boldsymbol{U}\cdot\nabla\boldsymbol{U}}_{\nabla\cdot(\boldsymbol{U}\boldsymbol{U})} \;=\; -\frac{1}{\rho}\nabla P + \nu\underbrace{\nabla\cdot\nabla\boldsymbol{U}}_{\nabla^2\boldsymbol{U}}$$

$$\frac{\partial U_j}{\partial t} + \underbrace{U_i\frac{\partial U_j}{\partial x_i}}_{\frac{\partial U_i U_j}{\partial x_i}} \;=\; -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu\frac{\partial^2 U_j}{\partial x_i^2}$$